TOPIC 2.7: GENERAL RESULTS OF POLYNOMIAL EQUATIONS

PERFORMANCE OBJECTIVES
Students should be able to:

- use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial
- write the equation of least degree given the complex or real zeros of a polynomial
- find the product and sum of the roots of an n^{th} degree polynomial
- use the Complex Conjugate Theorem to identify missing zeros of a function

MATERIALS

Graphing calculator

STRATEGIES (This lesson may require two days.)

- Start the lesson with the following Do Now:
  Write the equations with rational coefficients that have the following real roots:

  (a) 2 and 3
  (b) 2 and $3 + \sqrt{2}$.

- Ask students what the relationship is between the number of roots and the type of equation that results from them. Elicit that (a) generates the quadratic equation $x^2 - 5x + 6 = 0$. Further elicit that based upon the rational coefficients of the equation, there are actually three roots, 2, $3 + \sqrt{2}$ and $3 - \sqrt{2}$ to part (b) of the do now. Elicit that the actual equation may be constructed by first forming: $(x - 2)(x - (3 - \sqrt{2}))(x - (3 + \sqrt{2})) = 0$ and multiplying out the result. The cubic equation that is formed by doing this is $x^3 - 8x^2 + 19x - 14 = 0$.

- After eliciting that an equation with 4 roots is quartic, and one with 5 roots is quintic, state the Fundamental Theorem of Algebra:

  Theorem I: A polynomial equation $P(x) = 0$ with degree n, $(n > 0)$, with complex coefficients has exactly *n* roots, provided that a double root is counted as two roots, a triple root is counted as three roots, etc.)

  Elicit also the next theorem regarding the presence of $3 - \sqrt{2}$ as a root:

  Theorem II: Suppose that $P(x)$ is a polynomial with rational coefficients, and *a* and *b* are rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a root of the equation $P(x) = 0$, then $a - \sqrt{b}$ is also a root. (Point out that if the restriction of the rational coefficients were removed, we could form a linear equation with only $3 - \sqrt{2}$ as the root. The resulting equation would be $x - 3 + \sqrt{2} = 0$.)

Challenge the class to predict what would happen if 2 + *i* were a root given the same circumstance as the preceding example. Elicit the next theorem:

Theorem III: If P(x) is a polynomial with real coefficients, and a + b*i* is an imaginary root of the equation P(x) = 0, then a – b*i* is also a root. (Point out that if the restriction of the real coefficients were removed, we could form a linear equation with only 2 + *i* as the root. The equation would be x – 2 – *i* = 0.)

A handout containing these three theorems, as well as the fourth and fifth ones, with illustrated examples, could be distributed at this point.

- To assess understanding of the previous three theorems, pose the following to the class: What is the least degree of an equation with rational coefficients whose roots are 2, 2, 2 – *i* and 2 + $\sqrt{5}$? Elicit that since 2 + *i* and 2 – $\sqrt{5}$ must also be roots by the previous two theorems, the equation with the least degree would be 6.

- In order to motivate the next theorem, pose the following example:

  (a) Use the graphing calculator to graph, $y = x^3 + 7x^2 - x + 3$.
  (b) How many real zeros does this function have?
  (c) Try to draw a cubic equation that has no real zeros

- Use the preceding example to elicit that it is impossible to draw a cubic equation that has no real zeros. Further elicit that all cubic equations, no matter how they are graphed, must have at least one zero. Generalize this to a linear function L(x) or a quintic equation P(x). Summarize this as the following

  Theorem IV: If P(x) is a polynomial of odd degree with real coefficients, then the equation P(x) = 0 has at least one real zero.

- To introduce the last general theorem of polynomial equations, pose the following to the class:

  (a) What is the sum and product of the roots of $x^3 - 8x^2 + 19x - 14 = 0$?
  (b) Where do you think the coefficient of the x term, +19, comes from?
  (c) Where do you think the coefficient of the x^2 term, -8, comes from?

  Elicit the following

  Theorem V: For the equation $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 = 0$, with $a_n \neq 0$, the sum of the roots is $-\frac{a_{n-1}}{a_n}$ and the product of the roots is $\frac{a_0}{a_n}$ if *n* is even, and $-\frac{a_0}{a_n}$ if *n* is odd.

  So, for quadratic or quartic equations, the product of the roots is $\frac{a_0}{a_n}$ while for cubic or quintic equations, the product of the roots is $-\frac{a_0}{a_n}$.

To answer part (b) of the problem, lead the class through a demonstration such that when the sum of the roots is taken two at a time, we get 19. The coefficient of the ‘x’ term can be demonstrated by: $(2)(3 + \sqrt{2}) + (2)(3 - \sqrt{2}) + (3 + \sqrt{2})(3 - \sqrt{2}) = 19$. Point out that because of the conjugate pairs of the roots, the terms containing $\sqrt{2}$ all cancel each other. This will also happen when the roots appear in conjugate complex pairs.

If time permits, lead the class through the answer to part (c).

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