TOPIC 4.7: FORMING FUNCTIONS FROM VERBAL DESCRIPTIONS

PERFORMANCE OBJECTIVE:

Students will be able to:

- form a function in one variable from a verbal description
- use the graphing calculator to approximate a value for a graphic solution to a verbal problem

MATERIALS:

Graphing calculator, prepared overhead transparency

STRATEGIES: (This lesson may require two days to cover a sufficient examples.)

- Pose the following problem as a Do Now: A pile of sand is in the shape of a cone with a diameter that is twice its height. Sketch the diagram. Express the volume V as a function of the height of the pile. Elicit that since the volume of a cone is given by V = $\frac{1}{3}\pi^2\mathbf{h}$ and d = 2h, therefore 2r = 2h or h = r, we have V = $\frac{1}{3}\pi\mathbf{r}^3$. In this last equation, the volume is a function of the radius of the cone.

- An open top box with a square base is to be constructed from sheet metal in such a way that the completed box is made from 2 square yards of sheet metal. Express the volume of the box as a multivariable function of the base and width of the square base dimensions. Express the volume of the box as a single function of the width of the square base.

- Elicit from the class how to identify given information in a verbal problem and assign variables. In the example of the do now, the multivariable function for volume may be written as the formula for a rectangular solid: $V(w, h) = w^2h$. In order to write the volume as a function of the base, we use the fact that the surface area is two and that the surface area is computed by finding the area of the square base, w^2, and areas of the four vertical sides 4wh. The equation is $w^2 + 4wh = 2$. Since the do now asks that the volume be expressed as a function of just the width of the base alone, we eliminate the h from the volume formula above by solving the second equation above for h = $\frac{2-w^2}{4w}$. Substituting this into the $V(w,h) = w^2h$, we have V(w) = $\frac{2w-w^3}{4}$. In order to determine the domain of w for this example, consider the extreme cases of a rectangular solid, one having a very small value of w, and another have its largest value. The domain may be determined from examining the graph of V(w)=$\frac{2w-w^3}{4}$ since it serves as the mathematical model for this problem.

First, the width must be positive, therefore w > 0. Second, from the graph of V, if w > $\sqrt{2}$, then V < 0. Therefore, the domain is 0 < w < $\sqrt{2}$.

- Lead the class through a discussion of the steps involved in setting up and solving this verbal problem:
  (1) Draw and label a diagram (or review the given diagram). Be sure to identify the domain of the variable used.
  (2) Write a function or a multivariable function to describe what is being asked.
  (3) Identify the given mathematical information and identify relevant formulas that will be used for solving the problems. In some cases, some of these formulas will need to be combined to satisfy the conditions of the problem.
  (4) Solve the problem.
  (5) Check that the solution complies with the conditions given in the real world situation and that the answer lies in the domain of the variable.

- Pose a second problem:
  Water flows into a conical tank 100 cm wide and 250 cm deep at a rate of 40 cm^3/s.

  (a) Find the volume (V) of the water in the tank as a function of the height *h* of the water.
  (b) Represent *h* as a function of the time *t* that the water has been flowing into the empty tank. Remind students that the volume of a cone is V(r, h) = $\frac{1}{3}\pi r^2 h$. Because the problem asks for h as a function of time, the quantity of r in the volume formula will need to be replaced by h, the altitude of the cone. We get this from similar triangles and the proportion: $\frac{\mathbf{r}}{\mathbf{h}} = \frac{50}{250}$, or r = $\frac{h}{5}$. From this and the volume formula, we have the solution to (a): V(h) = $\frac{\pi}{75} h^3$. The volume of the water may be represented by the expression V = 40t, since water is flowing in at a rate of 40 cm^3/s. Finally, substituting 40t for volume, and solving the preceding equation for h, we have h(t) = $\sqrt[3]{\frac{3000t}{\pi}}$, an expression that answers step (2).

- Discuss at this point, the relation between the extreme values (minimum or maximum) of a function y = f(x) and the function y = $\sqrt{f(x)}$. In order to motivate this, pose the following example: Use the graphing calculator to find a local maximum for y = $x^3 - 4x$. Then use it to find the local maximum at $\sqrt{x^3 - 4x}$. Elicit that both of these functions have local maxima at x = -1.15, even though the functions have different y values at these points.
  Repeat the exploration with the following example: Find the local minimum of
  y = $-x^2 + 3x - 2$ and y = $\sqrt{-x^2 + 3x - 2}$ using the graphing calculator.
  Again elicit that both of these functions have local maxima at x = 1.5.

Summarize the activity by stating: If y = f(x) has a minimum or maximum at x_0, and x_0 is in the domain of y = $\sqrt{f(x)}$, then y = $\sqrt{f(x)}$ also has a minimum or maximum at x_0. (This theorem is used in the calculus course in minimum – maximum problems where setting the derivative of y = f(x) equal to zero is much easier that setting the derivative of y = $\sqrt{f(x)}$ equal to zero.)

- Pose a third example: Find the point on the parabola y = x^2 that is closest to the point (6, 2). Elicit that the distance formula needs to serve as the basis for the mathematical model. Further elicit that any point on the parabola may be represented as (x, x^2). The distance between these two points serves as the model, i.e., D(x) = $\sqrt{(\mathbf{x}-6)^2+(\mathbf{x}^2-2)^2}$. This model has a local minima at (1.78, 4.38) which can be found using the TI-83 graphing calculator by pressing [2nd], [TRACE], [3]. This means that when x = 1.78, the distance is minimum at 4.38. The point on the parabola closest to (6,2) is (1.78, 3.17).

Lesson plan by Prachee Chitnis and Shana Henry