<u>TOPIC 5.4: THE NUMBER e AND THE FUNCTION e^x</u>

<u>PERFORMANCE OBJECTIVES:</u>

Students will be able to:

- recognize that the number *e* can be approximated using the expression $(1+\frac{1}{n})^n$
- recognize, evaluate, and graph exponential functions with base *e*
- use exponential functions with base *e* to model and solve real-life problems
- apply the number *e* to solving problems involving interest rate
- see the relationship between *e* and compound interest
- use different methods to approximate *e*

<u>STRATEGIES</u>

- To begin the lesson, have this "Do Now" activity ready on a transparency or on the board:
  (1) Copy and complete the table by using a calculator. Explain in two or more sentences what is happening to the expression as the values for *n* increases.

n	$(1+\frac{1}{n})^n$
10	
100	
1000	
10,000	
100,000	

  (2) To explore the concept of *e* covered in this activity, ask students to graph $(1+\frac{1}{n})^n$ on their graphing calculator. Have students use the TABLE feature to investigate the behavior of the graph as *n* increases. Use TBLSET to set TblStart = 0 and to set ΔTbl = 1000. Challenge the students to investigate of the graph as n gets very large. The students should realize that the graph is approaching a constant number of approximately 2.718.

- Point out that the result, which we call *e*, is one of those special numbers in mathematics like π that appears often. Point out that *e* was first discovered by Leonhard Euler and was named in his honor. Furthermore, these numbers are called transcendental numbers, i.e., they can not occur as a root of any polynomial function having rational coefficients.

- *e* occurs in applications to psychology. It is related to the amount of knowledge that a person retains after a certain amount of time has passed. The Ebbinghaus Model for human memory gives the percent, *p*, of acquired knowledge that a person retains after an amount of time. The formula, $p = (100-a)e^{-bt}$, is where *t* is the time in weeks, and *a* and *b* vary from one person to another. If for a certain student, there are given values of a = 18 and b = 0.6, how much information will the student retain two weeks after learning a new topic? A solution to this type of problem is called an exponential decay problem.

- *e* occurs in applications when computing compound interest. In order to develop this concept, begin with quarterly compounding and the following problem: Denord received a total of $8000 in cash gifts for his college graduation. He is going to invest his money in a savings account so that he can buy a car after he graduates from college. A savings bank offers a statement savings account that pays 4.5% APR **compounded quarterly**. At this rate, what would the balance of the savings account be after five years? Since Denord's savings account is compounded quarterly, and the annual rate is 4.5% or .045, the quarterly rate is $\frac{.045}{4}$. In five years, there are 20 (5*4) quarters. Elicit that by using the growth formula, we have $P_{final} = 8000(1 + \frac{.045}{4})^{20}$ = $10,006.00. Have the class repeat the problem with daily compounding to get $P_{final} = 8000(1 + \frac{.045}{365})^{1825}$ = $10,018.44. The 4.5% interest needs to be divided into 365 to calculate the daily interest and there are 365*5 = 1825 days in five years.

- Suppose the previous problem is compounded continuously instead of daily. This means that at each instant, the interest is computed and added to the account. The unit of time is now something that is getting smaller and smaller without limit. We will need to express this with limits. $P_{final} = 8000\,(1 + \frac{.045}{n})^{n \bullet 5}$. Direct substitution of an infinite value for n is impossible. We can rewrite the expression as $8000\,(1+\frac{1}{\frac{n}{.045}})^{\frac{n}{.045}\bullet(.045)\bullet 5}$. Isolating part of this expression as follows we have: $8000[\,(1+\frac{1}{\frac{n}{.045}})^{\frac{n}{.045}}\,]^{(.045)(5)}$. However the part of this expression in brackets is very similar to the do now, except for the denominator and exponent. As *n* gets very large, the expression in brackets becomes *e*. To compute the interest we need $P_{final} = 8000\, e^{(.045)(5)}$ = $10,018.58, an amount not too different from the previous daily compounding example. Summarize this lesson by writing the formula for continuously compounded interest as $P_{final} = P_0\, e^{r\,t}$, where P_0 is the original amount of principal, *r* is the interest rate, and *t* is the time invested.

- Pose the following to the class: Graph $y = e^x$ on your calculator and describe what family of graphs $y = e^x$ most closely resemble. Elicit $y = 2^x$, $y = 3^x$ and that the graph of $y = e^x$ lies between them, yet, all three pass through the point (0, 1).

- Explain how interest compounded **monthly** and compounded **continuously** are different. If the interest rates were the same, would you choose a savings account that compounded interest monthly or continuously? Why?

- *e* appears in the theory of population growth. A population of insects is growing in such a way that the number in the population *t* days from now is given by the formula $P = 4000e^{0.02t}$. Pose the question, "How large will the population be in one week?" Elicit that by substituting t = 7 into the formula, we can answer the question.

- Have the class find the value of the first 7 terms in the series: $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}$.
  Explain that this series is another way to arrive at *e* and was discovered by a mathematician called Taylor. It is an approximation of e = 2.71805555.

Lesson plan by Denord Rerrie and Stacey Lehrer

Compound Interest

Interest period	% growth each period	Growth factor each period	Amount
Annually	12%	$1+\frac{0.12}{1}$	$1.12^1 = 1.12$
Semiannually	6%	$1+\frac{0.12}{2}$	$1.06^2 = 1.1236$
Quarterly	3%	$1+\frac{0.12}{4}$	$1.03^4 = 1.1255$
Monthly	1%	$1+\frac{0.12}{12}$	$1.01^{12}=1.1268$
Daily 360 days	$\frac{12}{360}\%$	$1+\frac{0.12}{360}$	$(1+\frac{0.12}{360})^{360} \approx 1.1275$
k times per year	$\frac{12}{\mathbf{k}}\%$	$1+\frac{0.12}{\mathbf{k}}$	$(1+\frac{0.12}{\mathbf{k}})^{\mathbf{k}}$