TOPIC 5.7: EXPONENTIAL EQUATIONS; CHANGING BASES

PERFORMANCE OBJECTIVES

Students will be able to:

- solve exponential equations having bases that can not be made the same without logarithms
- change logarithms from one base to another

MATERIALS

Graphing calculator, transparencies

STRATEGIES

- Pose the following example to get the lesson started:
  Solve the equation $3^x = 12$ using (a) common logs and (b) using natural logs.

  Elicit and review that if the bases of an exponential equation can not be made the same, that logarithms must be used. We convert 3 to $10^{.4771}$ and 12 to $10^{1.0792}$ by using common logs. Point out to the class that by using logs, the same base that we are using is the base of the common log 10. Solving this equation we get x = 2.262. We streamline the solution by taking the log of both sides to get: $x = \frac{\log 12}{\log 3} = 2.262$.

  In order to solve (b) of the do now, we use natural logs. We convert 3 to $e^{1.0986}$ and 12 to $e^{2.4849}$. Solving this equation we get x = 2.262. We can streamline this solution too by taking the ln of both sides of the equation to get x ln 3 = ln 12. Solving for x we get 2.262. Challenge the class to explain why log 12 = 1.0792 and ln 12 = 2.4849. Elicit that since 10 is bigger than *e*, it takes a smaller number to raise 10 to get 24 than it takes *e* to do the same thing. Moreover, 12 is between 10^1 and 10^2 so log 12 is between 1 and 2. On the other hand, 12 is between e^2 and e^3, so ln 12 is between 2 and 3.

- In 1992, 6.7 billion people inhabited the earth. If the population is growing at 1.35% per year, estimate the year when the population will reach 12 billion. Elicit that by using the growth model, we have $P(t) = P_0 (1+ r)^t$. Also elicit that by substituting all of the information into this equation, we have $12 = 6.7(1 + .0135)^t$. This produces an exponential equation (after dividing both sides by 6.7) of $1.7910 = 1.0135^t$. Since the bases can not be made equal to each other, elicit that we may use either common logs or natural logs to solve the equation.

Elicit that we solve this equation the same way as we did the do now problem:
log 1.7910 = t log 1.0135. This yields a solution of approximately 44 years. So 44 years after 1992, in the year 2036, 12 billion people will inhabit the earth.

- The medial summary of the lesson should list the steps required to solve an exponential equation where the bases can not be made the same except through logarithms.
  (1) Write the equation of $A(t) = A_0 (1+ r)^t$ and substitute the information into this model.
  (2) When the equation is in the form of $a = b^x$ and *a* and *b* can not be expressed as in the same integer base, take the logarithm of both sides and solve the resulting equation.

- Introduce the continuous compounding formula that will involve *e* and require the student to take the natural log of both sides. Suppose you invest P dollars at an annual rate of 7% compounded daily. (a) How long does it take to increase your investment by 60%? (b) To triple your money? Elicit that because of compounding daily, we use the model: $P(t) = Pe^{rt}$ In order to answer (a) where the investment is increased by 60%, we represent this amount by P + .60P = 1.6P. Substituting into the equation we have $Pe^{.07t} = 1.6P$. Dividing by P we have $e^{.07t} = 1.6$, an exponential equation that requires the use of natural logarithms to solve because of the base *e*. Taking the natural log of both sides gives $\ln e^{.07t} = \ln 1.6$. This results in $.07t \bullet \ln e = \ln 1.6$ or simply $.07t = \ln 1.6$. Solving we have t = 6.7 years. So the investment will require about that much time to increase by 60%. In order to answer (b) we have to let 3P represent the tripled initial investment P. The equation is $Pe^{0.07t} = 3P$. After division by P, we have $e^{0.07t} = 3$. Taking the ln of both sides we have $0.07t = \ln 3$ or t = 15.7. Therefore it will take about 15.7 years to triple your money.

- Summarize the concepts presented in this lesson by posing the following example:
  How long will an investment take to double in value if the interest rate is *r*%?
  Using our model, we have $Pe^{.01rt} = 2P$. Solving, we have $e^{.01rt} = 2$ and consequently, $.01rt = \ln 2$. This yields a solution of $t = \frac{69.3}{r}$ years. This has often been generalized into "the rule of 72." To use this rule, divide the interest rate into 72 to find out in how many years your investment will double. At 8%, the investment will double in 9 years.

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