TOPIC 6.3: ELLIPSES

PERFORMANCE OBJECTIVES

Students will be able to:

- find the equations of ellipses given their intercepts or foci
- graph ellipses given their equations
- define the major and minor axis and foci of an ellipse

MATERIALS

Graphing calculator, graph paper, string, tape, transparencies

STRATEGY

- Begin the lesson with the following Do Now:
  On the graphing calculator, graph the following on the same set of axes:
  $$y = \pm\sqrt{9 - \frac{9}{16}x^2}$$
  (a) What is the shape of the graph?
  (b) What are the x-intercepts?
  (c) What are the y-intercepts?
  (d) What is the equation written without the radical.

- From the do now, elicit that the equation is that of an ellipse and that the x-intercepts are ± 4 and the y-intercepts are ± 3. Sketch the graph on the board. Do the following activity in groups: Give each group a sheet of graph paper, a piece of string and two small pieces of tape. Ask the students to tape the piece of string on two points on the same line, leaving some slack in the string. Have the students pull the string taut with a pencil and draw a curve around the top and the bottom. This can also be done on the front board with a longer string, possibly a sneaker or shoelace.

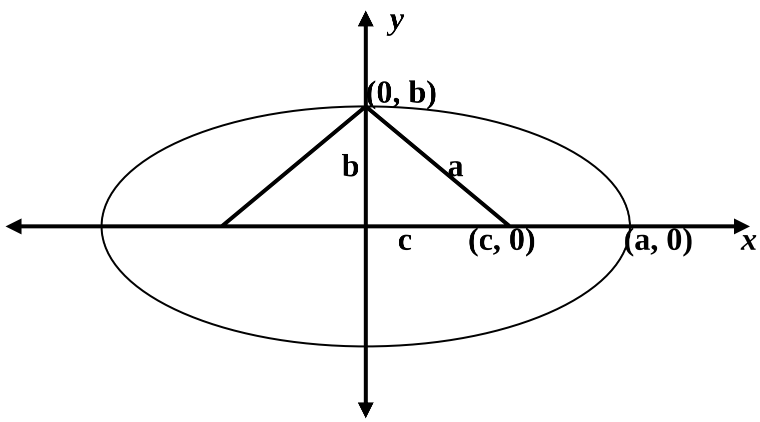
- Ask the students to label the two endpoints on the x-axis as (a, 0) and (-a, 0) and the two endpoints on the y-axis as (0, b) and (0, -b). Explain to the students that ellipses have two axis of symmetry called the major axis and the minor axis. The endpoints of the major axis are called the vertices of the ellipse. The points represented by the tape are called the foci (plural of focus) of the ellipse. Ask the students to label these points F_1(c, 0) and F_2(-c, 0).

- Ask the students to express the length of the major axis algebraically. Elicit the answer, 2a. Then ask the students to stretch the string so that it represents a point at (a, 0). Challenge the students to express the length of the string algebraically. Elicit that it is the sum of the segments formed from F_2 to the point and F_1 to the point and also that this sum is equal to 2a.

- Elicit and then write the following definition: If F_1 (c, 0) and F_2 (-c, 0) are two fixed points in the plane and *a* is constant, then the set of all points P in the plane such that $PF_1 + PF_2 = 2a$

is an ellipse. Derive the equation for the ellipse with foci F_1 (c, 0) and F_2 (-c, 0) by choosing any point on the ellipse P(x, y). Express PF_1 and PF_2 in terms of *x*, *y* and *c*, using the distance formula (see the page 79.)

- Pose the following to the class: Graph the equation of the following ellipse and find the coordinates of the vertices and foci: $\frac{x^2}{36} + \frac{y^2}{16} = 1$. The x-intercepts are (± 6, 0) and the y-intercepts are (0, ± 4). In order to find the foci's coordinates, we use the following equation: $b^2 = a^2 - c^2$. This equation is based on the right triangle formed inside the ellipse whose vertices are (0, 0), (0, b), and (c, 0) as shown in the diagram below. The hypotenuse of this triangle has a length of *a*, based on the definition of an ellipse. Using this, the coordinates of the foci are $(\pm\sqrt{20}, 0)$.



- Challenge the class to write the equation of an ellipse whose center is at the origin, one vertex is at (0, 5) and one focus is at (0, 2). Elicit that the major axis is the y-axis since the foci are located on it. Sketching the ellipse and the right triangle inside the ellipse whose vertices are (0, 0), (0, 2), and (a, 0) and the x-intercept of the graph, we have that $5^2 - 2^2 = 21$. The value of *a* is $\sqrt{21}$ and the equation of the ellipse in standard form is $\frac{x^2}{21} + \frac{y^2}{4} = 1$.

- Ask the students to graph the following equation on a set of axes: $\frac{(x-5)^2}{36} + \frac{y^2}{16} = 1$. What happens to the graph of an ellipse when x is replaced by (x – 5)? Elicit that there is a translation of the ellipse 5 units to the right. The new "vertices" are located at (11, 0), (-1, 0), (5, 4), and (5, -4). The new foci are located at $(5 + \sqrt{20})$ and $(5 - \sqrt{20})$.

Lesson plan by Tracy Saltwick, Steve Rossi and Maritza Rosado

Derivation of the Standard Form of an Ellipse

The Locus Definition of an Ellipse is the set of all P (x, y) in the plane such that the sum of the distances from two fixed points (called foci) is constant. To derive the equation for the ellipse with foci F_1 (c, 0) and F_2 (-c, 0), choose any point on the ellipse P(x, y).

$$| PF_1 + PF_2 | = \text{constant } k$$

Express PF_1 and PF_2 in terms of x, y and c, using the distance formula, as follows:

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = \text{constant } k$$

If we select the point on the ellipse (a, 0) and apply this definition, we see that the sum of the distance from (c, 0) to (a, 0) and back to (-c, 0) is actually 2a. So now we have:

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

Rewriting this equation, we have: $\sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$

Squaring both sides, we have:

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2$$

Simplifying by combining like terms, we have: $4cx = 4a^2 - 4a\sqrt{(x-c)^2+y^2}$

Divide by 4 and rearrange terms: $cx - a^2 = -a\sqrt{(x-c)^2+y^2}$

Square again: $c^2x^2 - 2a^2cx + a^4 = a^2 [(x - c)^2 + y^2]$

Simplify the right side of the equation: $c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$

Simplifying and rearranging terms: $a^4 - a^2c^2 = a^2x^2 - c^2x^2 + a^2y^2$

Factoring, this expression becomes: $a^2(a^2 - c^2) = (a^2 - c^2)x^2 + a^2y^2$

Since a must be greater than c, substitute $b^2 = a^2 - c^2$: $a^2b^2 = b^2x^2 + a^2y^2$

Dividing by a^2b^2, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$