TOPIC 6.4: HYPERBOLAS

PERFORMANCE OBJECTIVES

Students will be able to:

- find an equation of a hyperbola given vertices and foci or asymptotes
- graph a hyperbola given its equation
- identify the vertices, foci, asymptotes, and center of a hyperbola
- give the domain and range of a hyperbola

MATERIALS

Graphing calculator, overhead graphing calculator, overhead projector.

STRATEGIES

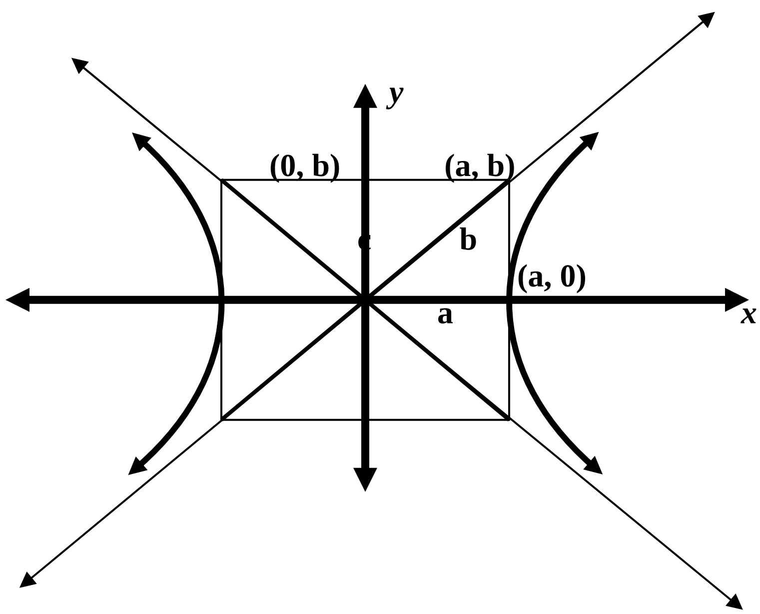
- Introduce hyperbolas to the class by having them use their graphing calculators to graph: $y = \sqrt{\frac{\mathbf{x}^2}{4} - 9}$ and $y = -\sqrt{\frac{\mathbf{x}^2}{4} - 9}$. Ensure that the window is correctly set so that all four quadrants can be seen. After the students each have their graphs, ask them what they see and what they think the equations are that represent these graphs. Identify the graph as a hyperbola and challenge them to write the equation in a standard form that closely resembles an ellipse. Elicit a form of $\frac{x^2}{4} - y^2 = 9$, an equation that may be obtained by squaring each side and rearranging terms. Dividing each term by 9, we have $\frac{x^2}{36} - \frac{y^2}{9} = 1$. Should students suggest that the graph is two parabolas, have them graph $y = \pm\sqrt{\mathbf{x} - 6}$, an east facing parabola whose vertex is (6, 0).

- Compare the x intercepts of the graph with the numbers in the equations and ask which number in the equation comes from the x-intercepts of the hyperbola? Notice that the x-intercept a of the hyperbola is the square root of the number under the x^2. Call this number a^2. Similarly, name b^2 the number under the y^2.

- Have the students then graph the equations $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$ on the same graph that the hyperbola was on. Do these two lines and the hyperbola ever intersect? How close do the lines get to the hyperbolas? The two lines that the hyperbola gets very close to, but never touches, are called asymptotes. Have the students graph the equation xy = 10. What are the asymptotes of this graph? What are the equations for the asymptotes of the graph? Elicit that by setting xy = 0, we get x = 0 and y = 0, the asymptotes. Similarly, by setting the left side of the equation in standard form equal to zero, the asymptotes of a hyperbola can be found. Setting $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, and solving for y, we get $y = \pm \frac{b}{a}x$.

- To find the x-intercepts of our graph with the numbers in the equations, take the square root of the number under the x^2, represented by a^2. The coordinates of the x-intercepts are then, (0, a) and (0, -a). The number under the y^2 is b^2. There are no y-intercepts to this graph. In general, the term that follows the minus sign in the equation has no y intercepts. If the standard form of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is rewritten as $\frac{x^2}{a^2} + \frac{y^2}{-(b)^2} = 1$, we see that the y intercepts are imaginary.

- The locus definition of the hyperbola is the set of all points P(x, y) such that $|PF_1 - PF_2| = 2a$ where F_1 and F_2 are the focal points of the hyperbola and *a* is the positive axis intercept of the hyperbola, as it was above. If we were to derive the equation for a hyperbola from this definition we get the equation $b^2 = c^2 - a^2$. This equation is based upon the right triangle formed in the rectangle, as shown in the diagram below with whose vertices are (0, 0), (a, 0) and (a, b). Use this to find c, and consequently, the coordinates of the focus points of our hyperbola.



- As a medial summary, have the students graph the hyperbola $\frac{y^2}{9} - \frac{x^2}{36} = 1$, either by using their graphing calculator or by sketching it on graph paper. Have them identify the foci and asymptotes of the graph, as well as the axis intercepts or vertices of the hyperbola. (The foci are at $\pm 3\sqrt{5}$. The asymptotes are $y = \pm\frac{1}{2}x$. The y-intercepts are at ± 3.) Have the students write the list of steps that are needed to graph a hyperbola and identify its intercepts, foci and asymptotes as a summary.

- Have the students find an equation for the hyperbola that has a vertex at (0, -12) and a focus at (0, -13). Remind the students that this will require them to work backwards to find an equation, where previously we worked forwards to get a graph from the equation. Have the students graph this on their calculators to make sure they have gotten the correct equation. The equation for this example is $\frac{y^2}{144} - \frac{x^2}{25} = 1$.

- As a final summary, have the students graph the following equation on their calculators: $\frac{(x-6)^2}{36}$ - $\frac{(y-8)^2}{64}$ = 1. Where is the center located? What did subtracting the 6 and 8 from the x and y, respectively, do to the graph? In our other graphs, the center of our hyperbolas was at (0,0) but where is the center of this graph? What are the coordinates of the foci and what are the equations of the asymptotes? Elicit that the foci are (16, 8), (-4, 8), the asymptotes are y = $\frac{4}{3}x$ and y = $-\frac{4}{3}x + 16$ with the center at (6, 8).

Lesson Plan by Rebecca Perkins, Lisa Gretano and Shawna Wan

Derivation of the Hyperbola Equation:

Locus Definition: The set of all P (x, y) in the plane such that $|PF_1 - PF_2| = 2a$, where F_1 (c, 0) and F_2 (-c, 0) are the foci of the hyperbola.

$$|PF_1 - PF_2| = 2a$$

The distance from any point P to the first focus point is obtained by using the distance formula. Similarly you get the distance from P to the second focus point. We replace these into the above equation to get:

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a$$

Rewriting this equation:

$$-\sqrt{(x+c)^2+y^2} = \pm 2a - \sqrt{(x-c)^2+y^2}$$

Squaring both sides of this equation we get:

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2$$

Canceling and combining like terms we have: $4cx - 4a^2 = -4\sqrt{(x-c)^2+y^2}$

Dividing each term by 4: $cx - a^2 = -a\sqrt{(x-c)^2+y^2}$

Squaring both sides again: $c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2)$

or $c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$

Canceling the $-2a^2cx$ from both sides: $c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$

Rewriting we have: $a^4 - a^2c^2 = a^2x^2 - c^2x^2 + a^2y^2$

Factoring we get: $a^2(a^2 - c^2) = (a^2 - c^2)x^2 + a^2y^2$

$c > a$, so substitute b^2 for $c^2 - a^2$ to get: $-a^2b^2 = -b^2x^2 + a^2y^2$

Dividing through by $-a^2b^2$ we have: $1 = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$