

TOPIC 6.7: A NEW LOOK AT CONIC SECTIONS

PERFORMANCE OBJECTIVES

Students will be able to:

- define eccentricity as a ratio
- define conic sections in terms of eccentricity
- classify a conic section knowing only its equation

MATERIALS

Overhead projector, transparencies

STRATEGIES (This lesson is optional and may be deleted without loss of continuity.)

- In order to get the lesson started, assign the following Do Now to the class:
 - (a) Sketch the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.
 - (b) Determine if the following points are points on the ellipse $P_1(0, -3)$, $P_2(4, \frac{9}{5})$, and $P_3(3, \frac{12}{5})$.
 - (c) Find the focus on the positive x-axis.
 - (d) Sketch on the same diagram a vertical line whose equation is $x = \frac{25}{4}$.
 - (e) Compute the ratios $\frac{P_1F}{P_1D}$, $\frac{P_2F}{P_2D}$, and $\frac{P_3F}{P_3D}$, where P_1D , P_2D and P_3D are the perpendicular distances from the points P_1 , P_2 , P_3 to line D , whose equation is $x = \frac{25}{4}$.
- Explain that this lesson explores another way of defining conic sections using eccentricity, denoted by the letter e . In the do now, the eccentricity of the ellipse is $\frac{4}{5}$. In general, let F be a fixed point not on line d . Let F (the focus) be a fixed point not on a fixed line d (the directrix). Let P be a point in the plane of d and F , and let PD be the perpendicular distance from P to d . Consider the set of points for which the ratio $PF:PD$ is the constant e (not to be confused with the transcendental number $e \sim 2.718$). When we draw the set of all points equidistant from F and d we have a parabola. Elicit from students the ratio $PF:PD$ is 1:1. We can define a parabola as a conic section whose eccentricity is 1.

Challenge the class to identify what would happen if the eccentricity of a conic section were a positive number less than 1. Draw a diagram on the board and elicit responses. This set of points is:

- (1) an ellipse if $0 < e < 1$
- (2) a parabola if $e = 1$ and
- (3) a hyperbola if $e > 1$.

This suggests for the first time that an ellipse and a hyperbola both have a directrix.

- Use the hyperbola, $\frac{y^2}{4} - \frac{x^2}{12} = 1$, with focus (0, 4) and directrix $y = 1$ and show that the eccentricity is 1. Elicit that this result is consistent with the concept that an eccentricity greater than 1 is for the hyperbola.
- Review with the class the characteristics that make a conic section's equation a circle, an ellipse, a hyperbola, or a parabola. Define the general equation for conic sections to be: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Explain to the class that the conic section having an xy term will be left to a more advanced course. However, we will be able to classify them at this point. Elicit:
 1. If $A = C$ and $B = 0$, we have a circle
 2. If $A \neq C$, and $B = 0$, and A and C have the same sign, we have an ellipse
 3. If $A \neq C$, and $B = 0$, and A and C have different signs, we have a hyperbola.
 4. If either $A = 0$ or $C = 0$, and $B = 0$, we have a parabola.
- Eccentricity is also defined for an ellipse and hyperbola as the ratio $\frac{c}{a}$, where c is the distance from the center to the focus and a is the distance from the center to the vertex. Explain that in an ellipse since $0 < c < a$, you can divide by a to show that $0 < e < 1$. When e is closer to zero, the two foci are near the center of the ellipse, and the ellipse looks closer in shape to a circle. When the eccentricity is closer to 1, it is more elongated.
- Elicit that in a hyperbola, the eccentricity, $e = \frac{c}{a}$, is a positive number greater than 1 since $c > a$. Pose the following problem: Write the equation of a hyperbola, with center at (-3, 1), a focus at (2, 1) and eccentricity $\frac{5}{4}$. Elicit that by using the given information, we have $c = 5$, because the distance between the center (-3, 1) and the focus (2, 1) is 5. From this we have, $a = 4$, and $a^2 = 16$. Using $b^2 = c^2 - a^2$ we have $b^2 = 9$. So the hyperbola's equation is $\frac{(x+3)^2}{16} - \frac{(y-1)^2}{9} = 1$.
- If the xy term is present, we can determine the type of conic by the following rule:
 - (1) If $B^2 - 4AC < 0$ then the graph is an ellipse (or a circle if $A = C$ and $B = 0$).
 - (2) If $B^2 - 4AC = 0$ the graph is a parabola.

(3) If $B^2 - 4AC > 0$ the graph is a hyperbola

(The derivation is beyond the scope of the course.)

Ask the students to identify the following:

(a) $x^2 - 3xy + 2y^2 + 2x - y + 6 = 0$ (Hyperbola)

(b) $x^2 - 6xy + 9y^2 + x - y - 1 = 0$ (Parabola)

(c) $144x^2 + 144y^2 - 216x + 96y - 47 = 0$ (Circle)

Transform the equations of this lesson into this eccentricity standard form and verify that the classification is consistent with the previous results.

Lesson plan by Chris Erchull and Prachee Chitnis