

TOPIC 7.6: INVERSE TRIGONOMETRIC FUNCTIONS

PERFORMANCE OBJECTIVES

The student should be able to:

- evaluate inverse trigonometric functions
- evaluate compositions of trigonometric functions
- find the principal value of trigonometric functions
- write equations for inverses of trigonometric functions
- graph inverses of trigonometric functions
- apply inverse trigonometric functions to solving real-world problems

MATERIALS

Graphing calculator

STRATEGIES

- To motivate the lesson, start with the following "do-now": using your graphing calculator, graph the following trigonometric equations in the same viewing rectangle, $y = \sin x$ and $y = \sin^{-1} x$. Describe the relationship between the graphs.
- If students have difficulty in seeing that one is the inverse of the other, ask them to graph $y = x$ to help them see that one graph is the reflection of the other graph in the line $y = x$. The entire reflection the graph of $y = \sin x$ over the line $y = x$ is not present on the graph screen. Take the opportunity to draw the graph of $y = \sin^{-1} x$ in its entirety on the board. Elicit that $y = \sin^{-1} x$ does not pass the vertical line test for functions. However, show that by restricting the domain of this inverse sine graph to $-1 \leq x \leq 1$ and the range to $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, it then passes the vertical line test for functions. We use the capital letter in the notation $y = \text{Sin}^{-1} x$ to denote that the range is restricted. Clarify that what the calculator draws is this graph with the restricted domain and range. The entire reflection of $y = \sin x$ is $y = \sin^{-1} x$.
- Summarize this discussion by writing the following functions on the board along with their respective domain and range. Given $y = \sin x$, the inverse Sine function is defined by the equation $y = \sin^{-1} x$ or $y = \arcsin x$. The domain of $y = \text{Sin}^{-1} x$ is $-1 \leq x \leq 1$ and range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Develop the graph $y = \text{Cos}^{-1} x$ in a similar manner and elicit that the domain of $y = \text{Cos}^{-1} x$ is $-1 \leq x \leq 1$ and that the range is $0 \leq y \leq \pi$. Finally, develop the graph of $y = \text{Tan}^{-1} x$ and elicit that its domain is all Real numbers and its range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Elicit that the reason that $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are removed from the range is because $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are asymptotes and these values are never reached in the graph.

- As a medial summary pose the following examples: (a) Evaluate $\text{Cos}^{-1}(-\frac{\sqrt{3}}{2})$. (b) Use a calculator to evaluate $\text{Tan}^{-1}(3.2)$. (c) Find the exact value of $\cos(\text{Cos}^{-1}\frac{\sqrt{2}}{2})$. Also use examples that require the use of quadrantal angles, such as (d) Evaluate $\text{Sin}^{-1}(-1)$. Using the 30-60-90 and the 45-45-90 degree triangles or the trigonometric graphs of Sine, Cosine and Tangent are required to find the exact values if the calculator is **not** used.
- Use a graphing calculator to graph $y = \text{Sin}^{-1}(\sin x)$. What are the domain and the range of this function? Explain why $\text{sin}^{-1}(\sin 4)$ does not equal 4. Now graph $y = \sin(\text{sin}^{-1} x)$ and determine the domain and range. Explain why $\sin(\text{sin}^{-1} 4)$ is not defined. In the discussion of this problem, review that the domain of $g(x)$ determines the domain of the composition of two functions $(f \circ g)(x)$. That is the reason why the two graphs are not the same and why the graph of $y = \sin(\text{Sin}^{-1}x)$ is only a restricted part of $y = \text{Sin}^{-1}(\sin x)$. Further explain that in evaluating $\text{Sin}^{-1}(\sin 4)$, we first find $\sin 4$ radians which equals $-.7568$, and then find $\text{Sin}^{-1}(-.7568)$ which equals $-.8584$. However, if we had used a value within the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, and found $\text{Sin}^{-1}(\sin \frac{\pi}{3})$, we would have $\frac{\pi}{3}$ or (1.0472) . Similarly, $\sin(\text{Sin}^{-1}(.3)) = .3$. The basic premise that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ still holds if the value of x is in the domain of both f and f^{-1} .
- (Optional) Malus' Law describes the amount of light transmitted through two polarizing lenses. If the axes of two lenses are at an angle of θ degrees, the intensity I of the light transmitted through them is determined by the equation $I = I_0 \cos^2\theta$, where I_0 is the intensity of light that shines on the lenses. At what angle should the axes be held so that one-half of the transmitted light passes through the lenses? SOLUTION: If one-half of the transmitted light passes through the lenses, then $I = \frac{1}{2} I_0$. Therefore, $I_0 \cos^2\theta = \frac{1}{2} I_0$, and by dividing each side by I_0 , we get $\cos^2\theta = \frac{1}{2}$. Taking the square root of both sides gives $\cos \theta = \pm \frac{\sqrt{2}}{2}$ which implies $\theta = \text{Cos}^{-1}(\frac{\sqrt{2}}{2})$ and $\theta = 45^\circ$.
- Pose the following writing questions to summarize the details of this lesson:
 - (a) Explain why the domain of the trigonometric functions must be restricted.
 - (b) How can we tell if the domain of a trigonometric function is restricted?
 - (c) Describe how to draw the graph of inverse trigonometric functions from the graph of the related trigonometric function.
- (Optional) Pose the following application: Highway curves are usually banked, or tilted inward, so that cars can negotiate the curve more safely. The proper banking angle θ , for a car making a turn of radius r feet at a velocity

v feet per second is given by $\tan \theta = \frac{v^2}{gr}$, where g is the acceleration due to gravity. (Assume that $g = 32 \frac{ft}{sec}$). A civil engineer is designing a curve with a radius of 1000 feet. If the speed limit on the curve will be 55mph, at what angle should the curve be banked?

Lead the class through the solution. First change the 55 miles per hour into feet per second.

We do this by as follows: $55 \frac{miles}{hour} = 55 \frac{5280 feet}{3600 sec} = 80.667 \frac{feet}{sec}$. Since we are given the

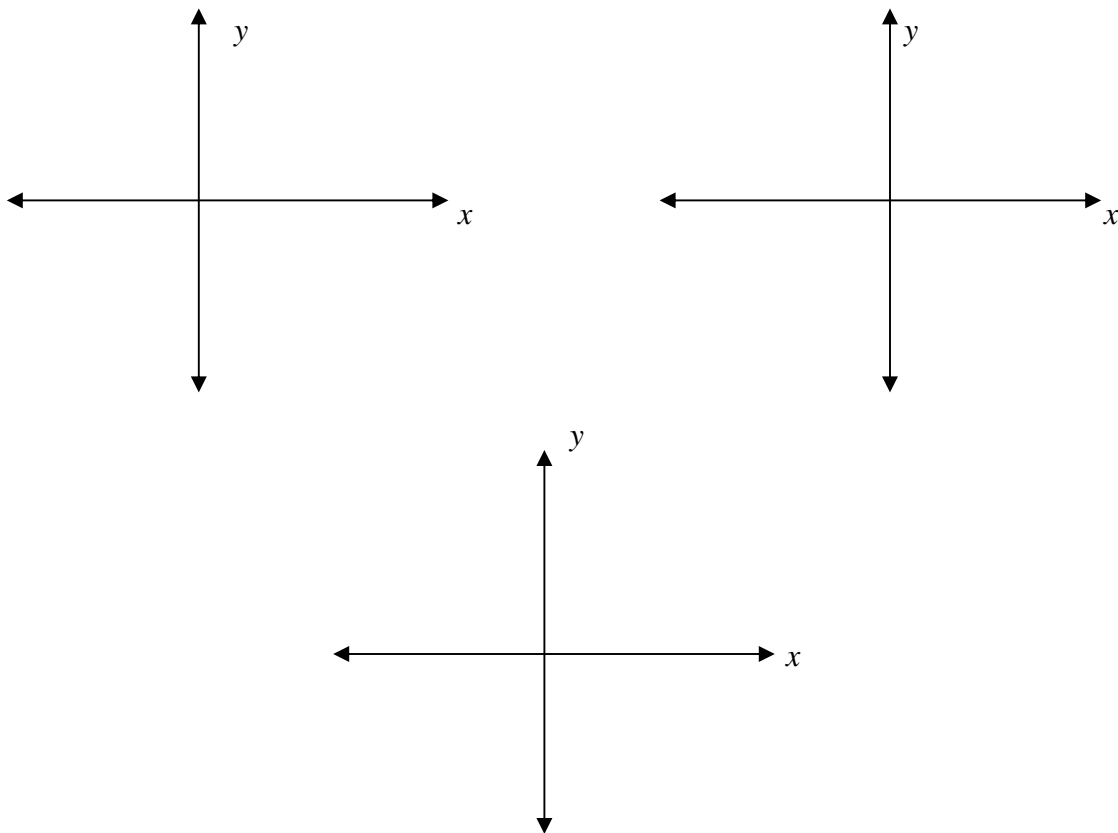
formula $\tan \theta = \frac{v^2}{gr}$, we have $\theta = \tan^{-1} \frac{v^2}{gr}$. Substituting 80.667 for v we have: $\theta =$

$$\tan^{-1} \left[\frac{(80.667 ft / s)^2}{(32 ft / s)(1000 ft)} \right] = \tan^{-1} (0.205031) = 11.5^\circ.$$

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THE INVERSE TRIGONOMETRIC FUNCTIONS

Function	$Y = \text{Sin}^{-1} x$	$Y = \text{Cos}^{-1} x$	$Y = \text{Tan}^{-1} x$
Domain			
Range			



Function	$Y = \text{Sin}^{-1} x$	$Y = \text{Cos}^{-1} x$	$Y = \text{Tan}^{-1} x$
Domain	$-1 \leq x \leq 1$	$-1 \leq x \leq 1$	All real numbers
Range	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$	$0 \leq y \leq \pi$	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$