TOPIC 8.4: RELATIONSHIPS AMONG THE FUNCTIONS

PERFORMANCE OBJECTIVES
Students will be able to:

- simplify trigonometric expressions
- state and apply the reciprocal identities
- state and apply the Pythagorean identities
- state and apply the relationship between f(-θ) and f(θ)
- state and apply the relation between a trigonometric function and its co-function
- prove trigonometric identities

MATERIALS

Graphing calculator, overhead transparencies, overhead projector

STRATEGIES

- Pose the following "Do Now" to the class:
  (1) Write each function in terms of sin θ and/or cos θ: (a) tan θ (b) sec θ (c) csc θ (d) cot θ
  (2) Write each of the following in terms of sin and/or cos and then simplify the expression:
  (a) $\frac{1}{\sin x \csc x}$ (b) tan x cos x

- From the "Do Now" review the trigonometric relationships and apply this knowledge to simplifying expressions. Elicit from the class that the first step in simplifying a trigonometric expression is to replace csc θ with $\frac{1}{\sin\theta}$ and sec θ with $\frac{1}{\cos\theta}$. These are two of the Reciprocal Identities. Another step in simplifying a trigonometric expression is to replace tan θ with $\frac{\sin\theta}{\cos\theta}$ and cot θ with $\frac{\cos\theta}{\sin\theta}$. These two relationships are the Quotient Identities. Have the class graph y = tan θ and y = $\frac{\sin\theta}{\cos\theta}$ and verify these are, in fact, the same graphs and that they have the same decimal values for each angle on the table.

- In order to motivate the next strategy in solving identities, pose the following: Prove that the following identity is true for all values of θ: $\frac{\sin\theta - \frac{1}{\sin\theta}}{\sin\theta + 1} = \frac{\sin\theta - 1}{\sin\theta}$. In this example, it is impossible to use either of the first set of strategies. Another strategy to consider is that whenever a complex fraction appears in an identity, simplify it by multiplying each term of the fraction by the LCD of the “little fractions.” In this case, elicit that the LCD of the “little fractions" is sin θ. Simplifying the complex fraction by this method yields $\frac{\sin^2\theta - 1}{\sin\theta + 1}$.

Factoring the numerator and simplifying the result, we have the left side of the identity equal to the right side and the problem is completed. In order to verify that the original equation is an identity, graph separately the left side and right side of the equation and elicit that both graphs are exactly the same.

- Illustrate the Pythagorean relationship of $\sin^2\theta + \cos^2\theta = 1$ by having the students graph $y = \sin^2\theta + \cos^2\theta$ on the graphing calculator. They will discover that the graph is the horizontal line, y = 1. Challenge the class to discover the relationship $1 + \tan^2\theta = \sec^2\theta$ by having them evaluate each expression for θ = 47°. Then, have them graph $y = 1 + \tan^2\theta$ and $y = \sec^2\theta$ on the same set of axes. (Note: Students will need to remember their reciprocal identities to be able to graph $y = \sec^2\theta$ on the graphing calculator.) Elicit that the two graphs are exactly identical. Have the students do the same for $1 + \cot^2\theta = \csc^2\theta$. Formalize the three Pythagorean Identities. A third strategy in proving identities is to use these Pythagorean Identities when the square of a trigonometric function appears in the identity. Challenge the class to prove the following:
  $\frac{1}{1-\sec A} + \frac{1}{1+\sec A} = -2\cot^2 A.$

- Pose the following to the class: Use a calculator to evaluate the following:
  (1) sin 50°, cos 40°　　(2) sin 25°, cos 65°
  (3) cos 11°, sin 79°　　(4) sin 83°, cos 7°

  Elicit conjectures regarding the special relationship between a function and its co-function. Formalize this by writing:
  1. sin θ = cos (90° - θ) and cos θ = sin (90° - θ)
  2. tan θ = cot (90° - θ) and cot θ = tan (90° - θ)
  3. sec θ = csc (90° - θ) and csc θ = sec (90° - θ)

  Use these identities to prove the following: sin x tan x + sin (90 – x) = sec x.

- Draw the unit circle on the board. Have students recall the coordinates of point P(x,y) in the first quadrant on this unit circle as (cos θ, sin θ). Further elicit that if this point is reflected in the y-axis, its image in the second quadrant is (-cos θ, sin θ). The next set of relationships among trigonometric functions can be drawn from the fact that the y value of both of these points is the same. Therefore, sin (-θ) = -sin θ. By reflecting the point P(x, y) in the x-axis, the image is (cos θ, -sin θ) and therefore, cos (-θ) = x = cos θ. Using the reciprocal and quotient relationships challenge the class to find the following relationships with negatives:
  (1) csc (-θ) = - csc θ　　(2) sec (-θ) = sec θ
  (3) tan (-θ) = - tan θ　　(4) cot (-θ) = - cot θ

  Use these to express sec (-250) as the function of a positive acute angle.

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